Optimal Detection of Blurred Edges

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Abstract

I develop a variational model for the detection of blurred edges. Though yielding a complicated general solution, the model gives insight into edge detection models previously proposed for step edges. In particular it shows that a previous IIR solution for step edges is correct even in the FIR case, and suggests why operators approximating the first derivative of a gaussian are successful, despite their suboptimality for step edge detection.

1 Introduction

Canny was the first to pose the image edge detection problem in variational terms [1]. He proposed three criteria for measuring edge detection performance – signal-to-noise ratio, edge localization, and the suppression of multiple responses – then optimized these jointly in a variational framework. His result, obtained numerically, was a convolutional operator that could be approximated well by the first derivative of a gaussian. Subsequently, several authors adopted the same approach, modifying one or more of Canny's criteria, or seeking an Infinite Impulse Response rather than a Finite Impulse Response solution. Many of these obtained operators that were similar in shape to Canny's. Representative examples are shown in Table 1. An exceptional case was the result obtained by Shen and Castan [2]. As is clear in the table, their operator is different in form from the other examples.

In this paper I consider a variational approach to the detection of blurred edges, showing how operators like Canny's and Shen and Castan's can both be obtained from the same set of initial constraints. In the process, I make assumptions common to work in optimal edge detection: edges are local phenomena demanding linear (convolutional) detection; two-dimensional solutions can be developed by considering one dimension first, then generalizing into orthogonal gradient-like estimators or families of oriented operators; noise is additive white and gaussian.

2 Model for blurred edge detection

Figure 1 shows the model used here for detection of blurred edges. It follows Shen and Castan's assumption that the solution will be a filter followed by differentiation.

For g(x), the negative exponential blur model, $g(x) = {}^{a}\!/_{2} \exp(-a|x|)$, will be used. The parameter a represents the amount of blur. We seek the positive half of f(x); the negative half will be its reflection. Doing this avoids constraining the solution to be smooth at 0.

In the FIR case, $f_L(x) = f_R(-x)$ with

$f(x) = 0$ for $|x| > W$.

In the IIR case, $f_L(x) = f_R(-x)$ with

$f(x) \rightarrow 0$ as $|x| \rightarrow$ infinity.

Other constraints are the "zero D.C. gain" constraint that the integral from 0 to infinity of $f_R(x)$ be ½, i.e. that the area under the whole of f(x) be 1; and that y(0) = T, which simply requires that the output at the edge point exceed some threshold. Note that although f(x) has a maximum at 0, it is not the case that

$$\frac{d}{dx} f(x)\Big|_{x=0} = 0$$

because the solution can lack C1 continuity at 0.

I minimize signal energy plus noise energy everywhere in y(x) except at x=0, considering first the interval [0,W). For the IIR case, W tends to infinity, which will be considered later.

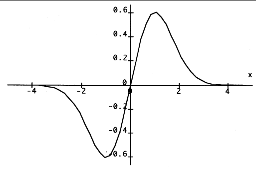
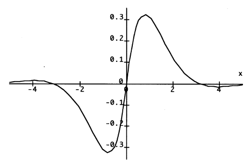
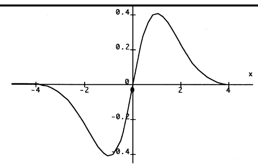
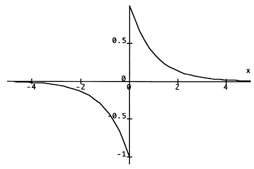
Authors	Optimization criteria	Analytic solution or approximation	Shape of the operator for typical parameter values
Canny[1]	Signal-to-noise ratio, Edge localization, Suppression of multiple responses.	$f(x) = \frac{x}{X^2} \exp(-\frac{x^2}{2\sigma^2})$	
Deriche [3]	As Canny, but for IIR solution.	$f(x) = -\exp(-\alpha \mid x \mid) \sin \alpha x$	
Sarkar, Boyer [4]	As Canny, but for IIR, and with modification of multiple response criterion.	$f(x) = A \exp(-\alpha \mid x \mid)(\cos(\varphi) - \cos(\beta \alpha x + \varphi))$	
Shen, Castan [2]	Operator will be smoothing followed by differentiation. Maximize response to step edge, Minimize noise in both filter and differential output.	$f(x) = \operatorname{sgn}(x) \frac{p^2}{2} \exp(-p \mid x \mid)$	

Table 1: Comparison of edge detectors derived by variational methods.

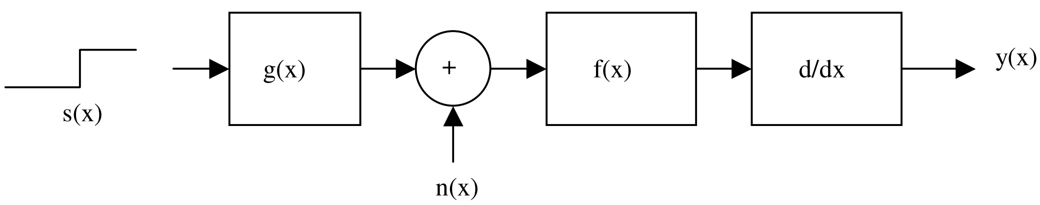


Figure 1: Model for edge detection: s(x) is a (one-dimensional) step edge of height A; g(x) is the blurring function; n(x) is additive gaussian noise; f(x) is the filter to be determined; d/dx is the differentiation operation; y(x) is the output signal.

3 Variational formulation

Signal energy in y(x) $= \int_0^W (\frac{d}{dx}(s(x) * g(x) * f(x))^2)dx = A^2 \int_0^W (g(x) * f(x))^2$

Noise energy in y(x) $= n^2 \int_0^W f'^2(x)dx$

Now let $h(x) = g(x) * f(x) = \frac{a}{2} e^{-a|x|} * f(x)$.

Then, because the inverse filter to the exponential is 1 - laplacian, we can write $f(x) = h(x) - \frac{1}{a^2} h''(x)$.

Therefore the total energy $= \int_0^W A^2 h^2(x) + n^2 (h'^2(x) - \frac{2}{a^2} h'(x)h'''(x) + \frac{1}{a^4} h'''^2(x))dx$

after substituting in for $g(x) * f(x)$ and $f'^2(x)$. Now, using R for the RMS SNR ($= A/n$) we obtain

Total energy $= n^2 \int_0^W R^2 h^2(x) + h'^2(x) - \frac{2}{a^2} h'(x)h'''(x) + \frac{1}{a^4} h'''^2(x)dx$

which we want to minimize subject to the constaint $\int_0^\infty f_R(x) = \frac{1}{2}$, or, in terms of $h(x)$:

$$\int_0^W h(x) - \frac{1}{a^2} h''(x)dx = \frac{1}{2}$$

We now form the criterion function J such that:

$$J = R^2 h^2(x) + h'^2(x) - \frac{2}{a^2} h'(x)h'''(x) + \frac{1}{a^4} h'''^2(x) - \lambda(h(x) - \frac{1}{a^2} h''(x))dx$$

where λ is a Lagrange multiplier.

The Euler - Lagrange necessary condition is :

$$\frac{\partial J}{\partial h} - \frac{d}{dx}\left(\frac{\partial J}{\partial h'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial J}{\partial h''}\right) - \frac{d^2}{dx^2}\left(\frac{\partial J}{\partial h'''}\right) = 0$$

which, after substitution and simplification, reduces to

$$R^2 h(x) - \frac{\lambda}{2} - h''(x) + \frac{2}{a^2} h''''(x) - \frac{1}{a^4} h''''''(x) = 0$$

The homogeneous differential equation has constant coefficients, so the solutions are of the form exp(αx). The characteristic equation is:

$$\alpha^6 - 2a^2\alpha^4 + a^4\alpha^2 - a^4 R^2 = 0$$

The roots of this equation are $\pm\delta, \pm\gamma \pm j\omega$, where δ, γ and ω are complicated. (Each is three lines in Maple format for the case where $27R^2 > 4a^2$)

The general solution to the differential equation is

$$h(x) = k_1 e^{\delta x} + k_2 e^{-\delta x} + e^{\gamma x}(k_3 \cos\omega x + k_4 \sin\omega x) + e^{-\gamma x}(k_5 \cos\omega x + k_6 \sin\omega x) + k_7$$

(the k_7 is because the original equation was inhomogeneous). Because of the relationship between f(x) and h(x), this immediately gives the general form of f(x) :

$$f(x) = c_1 e^{\delta x} + c_2 e^{-\delta x} + e^{\gamma x}(c_3 \cos\omega x + c_4 \sin\omega x) + e^{-\gamma x}(c_5 \cos\omega x + c_6 \sin\omega x) + c_7$$

4 Special cases

The final form for f(x) is very general. It depends on the parameters of blur, noise variance and the lagrange multiplier. By varying these parameters it is possible to obtain edge profiles similar to any of the earlier edge detectors. To gain further insight, we must therefore consider special (or, at least, restricted) cases.

In the IIR case, f(x) must tend to 0 as x tends to infinity, so the following solution is achieved:

$$f(x) = c_2 e^{-\delta x} + e^{-\gamma x}(c_5 \cos \omega x + c_6 \sin \omega x)$$

This is a blended combination of Sarkar and Boyer's result and Shen and Castan's result. With high blur, the second term predominates and the operator shape is similar to that in the third row of figure 1. As blur diminishes, the first term becomes more important and the shape moves towards that shown in the bottom row of figure 1.

Returning to the characteristic equation, we see that if a is very large (i.e. the edge is almost a perfect step), then, by division by a^4, it is clear that the optimal filter is simply a negative exponential:

$$f(x) = -c \exp(-a \mid x \mid)$$

This result is as true for the FIR case as the IIR case. That is, a truncated negative exponential is optimal for a step edge in noise. These results show that Shen and Castan's approach does indeed produce the optimal operator for step edges. This contradicts Canny's result because (a) he had an unnecessary f'(0)=0 constraint, and (b) his multiple-response criterion was inappropriate for step edges.

Experimentally I have confirmed Shen and Castan's finding that the negative exponential operator gives superior performance to the gaussian operator for detection of step edges. In line with the above theory, this is true for truncated operators too, provided they are several pels wide. However, for blurred edges, the negative exponential operator degrades very quickly.

In the general (FIR, blurred) case, the operator is, as already mentioned, very general. However, I have plotted it for various small values of a (i.e. moderate blur) and a range of noise levels. The resulting operator shape is almost always close to that of Canny, Deriche and Sarkar and Boyer. This suggests that these operators all work well for blurred edges. Moreover, because blur happens at many different scales, the best overall operator is going to be some kind of average – intuitively suggesting a gaussian shaping. Although a wide range of particular choices for a in the above formulation yield shapes close to a gaussian, the correct analytic process would be to integrate over a range of a, which I have not been able to do. Nonetheless, the results strongly suggest that the first derivative of a gaussian is a good operator for detection of blurred edges over a range of scales. Canny's operator's success suggests that although his multiple response criterion was incorrect for step edges, it is appropriate for preventing spurious responses to blurred edges.

5 Conclusion

The variational formulation developed in this paper for detection of blurred edges yields a very general result. This is reminiscent of (though worse than) earlier variational approaches to edge detection, where the form of the detector depends crucially on free parameters (such as the Lagrange multiplier). By restricting the solution to IIR or step-edge detection, it is possible to reduce the number of parameters and obtain insight into previous work. In particular, the step-edge detection case yields a negative exponential operator, and particular levels of blur yield operators that are blended combinations of negative exponential and windowed trigonometric operators. In appearance these operators are similar to earlier gaussian-smoothed operators.

References

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